Geometry
& Art
Geometry with eyes
SUMMATION
of the CONFERENCE

A tribute to the great pedagog Jan Amos Komenský (Comenius)
Part I - Pure mathematics
The golden ratio, originally

Look at this figure. A double square, its diagonal and its bisector. When you extend the bisector, it crosses the median line, horizontal. And this crossing point shows the golden ratio.

This is the first definition of golden ratio, the oldest. In the same time:
- A construction (the simplest)
- A definition (the easiest)
- A property, with angles:

\[\text{The great angle of the diagonal of a double square is two times the small angle of a golden ratio.}\]

This way of thinking is called « Geometry with eyes ». It developed a long time before calculation, before the writing and the concept of surface. Maybe during the late paleolithic, but we may in this case reinvent the concept of “paleo”, because \textit{geometry leads straight to the city}.

This figure has a prolongation (next page) and also a mathematical context. It doesn't emerge of nowhere, and it belong to a coherent whole that Thales and Pythagoras went to pick up in Egypt.
At first, let us consider a circumcircle, pointing in O - the center of the double square. This circle is passing to the crossing point I, with $AI = \varphi$.

The complementary segment to $\varphi$ of the diameter is $1/\varphi$. So: $\varphi + 1/\varphi = \sqrt{5}$

Another property appears: in a golden rectangle:

when a diagonal is horizontal, the other is the one of a double-square.

Albrecht Dürer uses this principle in his famous engraving « Melencolia § I ».

The grid

Geometry with eyes is built on a grid. This framework allows to construct, to demonstrate and to remember all the elements.

The queen figure of this practice is the modest triangle 3-4-5. Its bisectors are the natural diagonals of a simple, a double and a triple square. And the golden ratio is on the second bisector, between summit and incircle.
Another way to show the properties of this « sacred triangle » - so-called surveyors triangle when it worth much more. The sum of the orders (number of squares linked to each bisector) with the measure of the opposite sides is 6. This 6 is the base of the numeric development in esotericism. A dice has six faces, with the six integers from 1 to 6. The sum of the opposite faces is 7.

This figure has a great importance : it shows in a simple way that pre-euclidean geometry is not necessarily axiomatic and empirical. The side 5 is not a convention. It is possible, with similar triangles, to prove this assertion.
Four manifestations of $\phi$ in the triangle 3-4-5

1\textsuperscript{st} manifestation: as we have seen, $\phi$ is in the triangle. The golden rectangle $2 \times 2\phi$ is at its place, called "classic", riding on the bisector of order 2.

2. $\phi$ is the distance between the apex of the triangle and its inscribed circle.

2\textsuperscript{nd} manifestation: the first manifestation concerns the measurement, the second the angles. The bisectors of order 1 and 3 are the natural diagonals of a golden rectangle. This will lead to a general observation about symbolic.

3\textsuperscript{rd} manifestation: the classical rectangle is divided into an unlimited number of kites (which are in fact disguised golden rectangles). These forms build a golden spiral that converges to the point T of the hypotenuse, located one unit from the top of the triangle.
**4th manifestation**: a pentagram comes from the confrontation of the inscribed circle with its twin. This second circle puts its center two unities above the base of the triangle (side 4), and at a distance of $\phi$ from the side 3. This pentagram is inclined like the diagonal of a golden rectangle.

**Demonstrations**

The necessary elements in demonstration are very few. We can tell that they are the axioms of Thales. The evidences of similar triangles. The other aspects are very accessible. The process doesn’t need calculation, and the proofs by eyes are called « monstrations ».

**The sum of the angles of a triangle**

We just need to link three same triangles by their different angles at the same point O to show that the sum of the angles is $180^\circ$ (also denoted by $\pi$, or flat angle, or half a turn) triangles. Jean-Paul Guichard notices that there is even simpler: Olivier Keller relates in his book « An Archaeology of geometry », a series of Paleolithic carved friezes, on bone. These figures show a structure that allows the same observation.
The figures of sacred geometry

The hexagram

When a circle diameter 4 crosses a grid 4x4, the intersections show the two triangles of an hexagram. The figure of zodiac.

The vesica piscis

The vesica piscis is formed by two twin circles: the center of each one is on the other. Considered as sacred by the Pythagoreans, this pattern is the archaic and original symbol of Venus - long before she inherits all the qualities which overprint her real status.
The explicit vulva of the almond is modestly termed as “déïque” in french... Consequently, it represents the sacred feminine. However, in paleolithic times, the 3 with which the figure is linked (by the root) is not still revealed...

N.B. : The mandorla of Christ is never a vesica piscis. Other constraints define the almond, like a circumscribed rectangle with a ratio $\sqrt{3}$.

**Hexagon and Vesica Piscis**

Here is the figure head of the Byzantine grid of Andrei Rublev, which was used to build the "Holy Trinity". This figure is a diapason for the entire work.

![Hexagon and Vesica Piscis](image)

The internal hexagon of the hexagon takes the measure of the vesica piscis, whose almond is 2 in height. The circle of vesica is accordingly radius $2/\sqrt{3}$. The almonds of the other Vesica (45° inclined) are on the hexagon. NB: the Byzantine grid is not a trivial grid, even governed by $\phi$...

**Pentagon and Vesica Piscis**

![Pentagon and Vesica Piscis](image)

A tile is here $1/2$ \[ \varphi = (1 + \sqrt{5})/2 = 1/2 + \sqrt{5}/2 \]

8 point of 10 of the pentagram are on the Vesica Piscis.

So, the main figures of sacred geometry are naturally linked together.
Calculation and writing appear during neolithic, bringing a new breath to the geometry on the grid. This phenomenon concerns all the civilizations, but we might tell that Egyptians are more sentimental and artistic in front of Mesopotamians, more organized and abstract. Obviously they exchange their knowledge, but their quests are differentiated.

Babylonians will translate the experience of geometry into numbers. This new development will become Kabbalah. We investigate here the first step.

The figure of paragonal

In any triangle, the sum of the three angles equal π - or half turn or 180°. $\angle 1 + \angle 2 = 90° - \angle 3$ is available with any sequence of $\angle 1$, $\angle 2$ and $\angle 3$.

$\Rightarrow$ One of them can be a right angle.

When the triangle is right, there is a way to translate the previous property. If the blue bisector is fixed by the right angle, the others are linked by the equation $\angle 1 + \angle 2 = 90° - \angle 3$ with $\angle 3 = 45°$. So $\angle 2 = 45° - \angle 1$. 
In geometry, it means that we can find the second bisector from the first. This first bisector can be considered as the diagonal (DE) of a rectangle. This rectangle can turn by 45°, with the diagonal (EF) - right angle with (DE). And finally the line (DF) has the inclination of the second bisector. The French mathematician Raphaël Legoy has given the name of paragonale to this line, when we were studying the tablet Plimpton 322...

**Translation - from triple to pair**

The next figure helps to understand the right triangle in a particular way.
Nowadays, we are used to the Pythagoras theorem, but the « geometry with
eyes » doesn't need the concept of surface to understand the right triangle:
we think here through angles, as well as about the golden ratio.

This simple example, the triangle 21-20-29, shows all the equations of the
pairs (p,q) with p and q coprime numbers.

We have seen that the bisector coming from the vertex B of the triangle is
automatically linked to that one which goes to A, on the right (figure of
paragonal). Then the point B', symmetric to B with respect to this paragonal
(green) builds a segment B'O (red) which is right angle to the red bisector too,
BO. The proof is on the picture : the diagonals of two yellow and green
modules just to the left of O are right angle (and their paragonal is orthogonal
to OA).

This figure allows assumptions that other triangles will confirm.

The ratio of p and q here is 5/2.
It is irreducible.

Note that three modules are required to cover OB.
And 3 = q-p
We also note that two assembled modules cover OA
And 2 = p

From these assumptions, the measurements of the triangle are simple
calculation.

\[
\begin{align*}
BA &= B'O' + O'A = (q-p)q + p(q+p) = q^2 + p^2 \\
CA &= CO' + O'A = (q-p)p + p(q+p) = 2pq \\
CB &= (q-p)p + (q-p)q = q^2 + p^2
\end{align*}
\]

What manuals related as:
\[
\begin{align*}
a &= q^2 - p^2 \\
b &= 2pq \\
c &= q^2 + p^2
\end{align*}
\]

**The conjecture of Plimpton 322**

The famous tablet Plimpton 322 (18th Century BC) is already cracked by
mathematicians. Even the reconstitution of the missing lines. Under these
conditions, a simple calculation between the columns shows a series of prime
numbers, absolutely unexpected. This surprise deserves the name of
« conjecture of Plimpton ».

Let's keep our eyes on geometry.
Part II - Application in Art
Painters of Renaissance

We are used to say that in art the perspective system is the soul of progress during this period. For sure, this new way to manage the lines of the reality find their mathematical rules at that time. But besides this neutral system, in terms of symbolic, the artists of Renaissance continue to practice the older one: sacred geometry.

For who doesn't know what is composition, here is the simplest definition: The composition is a set of lines, associated with each other with a mathematical sense, that guide the final drawing. The pencil of the artist or the architect will search these geometrical figures, and these marks are like the wood of a formwork in the building: they are digested by the site during its progression, up to disappear. In the case of sacred geometry, the grid allows to translate the figures into numbers (it works also in the inverted sense). This brings the real meaning to the final work. The perspective system brings realism, but no symbolic meaning, by definition (we can not rely on any interpretation from the audience when the authors have no vocabulary to express their will).

In his « School of Athens », Raphael mixes the two system. The lines of perspective refer to the one of sacred geometry by their angles.
Eg the two white lines going down make an angle of 36° with the vertical. This
refers to the pentagram. The feeling of harmony doesn't depend on the one of
reality. This angle belongs to an other construction.

The measure of $\phi$ is given by Plato, and confirmed by the hand of Aristotle.

« Saint Michel and Satan » - 1518
The sacred geometry of Raphael is simple, and as ever with this painter,
terrifically efficient. The arrow, drawn by the geometry, is more effective than
the spear itself!
History of Art

La Grande Odalisque - 1814

Up to XIX\textsuperscript{th} Century we find traces of this culture. [This geometry with eyes is built on a grid, which allows to measure the forms. The numbers open to a symbolic translation in a human language]. In this work, by J.-D. Ingres, the great circle is diameter 3, measure of Celestial. The grid is, as usually, revealed by the triangle 3-4-5. And the side of the green square is $2\varphi$.

Eanna temple - Uruk IV - IV\textsuperscript{th} millennium B.C.

This practice invests mesopotamian architecture in early neolithic.
Plateau of Giza - 2500 years B.C.
Soon, Egypt shows the same skills

The famous tablet Plimpton 322, dating from 1800 y. B.C., is now explained as a list of pythagorean triplets, reduced to couples. These couples produce a list of primes, and the missing lines of the tablet respect the rule! The principe seems extendable, two values of the triplets becoming the couple of a next triplet. Pythagoreans gather the both influences, mesopotamian and egyptian. During centuries Greece is the crossroads of knowledge.
Sacred Geometry tries to escape to the Roman realism that Vitruve will express in his architectural treatise - 1st Century B.C., and it spreads in the Celtic world. The archaeologist Jean-Loup Flouest and mathematician Marc Bacault have shown the influence of Pythagoreans on the Celtic world.

Celtic phalera in Champagne, France
190 circles and arcs are needed to draw the elements of this object. Based on the numbers 8 and 27, elements fetishes of the numerology taught by Pythagoras, this object also shows the exchanges between the Latin scientific elite, and Celtic druids, who were considered at the time as Pythagoreans. The level of these mathematics requires four years of university studies.

The geometrical key of Germigny - France - 9th Century
At this period, a lot of artists and builders escape towards the west part of Europa, and they save their life but also their culture. In the “oratoire” of Germigny, they put the basics about the forms in a didactic scale.
The extreme solution will be once more the celtic world. The irish monks welcome this culture and “rewrite” the Holy Bible. By the way, the famous « Book of Kells » is known to take a great liberty with the text, as if the message had to be clear : geometry is the main purpose of this work.

Root of 3 and Phi ($\sqrt{3} & \varphi$)
The Madonna of Vladimir - 12th Century

In early middle ages, this culture finds a harbor in Byzantium. Icons and ceramics. We see here an accomplishment with the gift of the Patriarch of Constantinople to the great prince of Kiev in 1130. (The orthodox church separated from the catholic one in 1054). This quiet orthodox world is shaken by iconoclasm: between 730 and 787, then between 813 and 843.

The cathedral of Dol-de-Bretagne - France
mainly 13th and 14th Centuries

The France of cathedrals is maybe the result of this explosion - due to iconoclasm. A new idea is emerging in my researches: I was so obsessed by foreign influences that I didn't see that France was able to participate and to give something to this art, as well as northern Italy or Novgorod.
Conques - 11-12\textsuperscript{th} C - High school of art roman
Previous page - The heptagram of the tympanum
The composition of Conques is a perfect demonstration of the developments on a grid. Several systems link their layers on this same fame. Here the heptagram, remembering Byzantine knowledge. Huge file.

The key of the mandorla - a triple-square

\[ \mathcal{H} = \frac{1 + \sqrt{3}}{2} \]

The (unique case of a) lesson of Geometry in the letter G!
\( \mathcal{H} \) is here the value \((1+\sqrt{3})/2\), parent to the golden ratio. The figure is constituted of a circle at the top of an equilateral triangle.
The triangle is also used for measuring the small rectangle at the bottom. Algebraically, this means that $(\sqrt{3} - 1) = 2/(1 + \sqrt{3})$, which is not obvious a priori. But $(\sqrt{3} - 1)(\sqrt{3} + 1) = 3 - 1 = 2$. There is much more obvious. The ancients have demonstrated this with the eyes without going through the calculation. In other words, a $\sqrt{3}$ rectangle is the sum of a square plus a $H$ rectangle $[H = (1+\sqrt{3})/2]$

**Late middle ages - French altitude**

The House with the bell in Prague - Early 14th Century (delicious)

Le portrait de Charles VII (1450/55) - Jean Fouquet
Now we enter in a period of apotheosis for sacred geometry: the Renaissance. Three pieces embody this accomplishment: « The Holy Trinity » by Andreï Rublev, in 1420/28, « The birth of Venus » by Sandro Botticelli, in 1485, and « MELENQOLIA § I » by Albrecht Dürer, in 1514. The third belongs to a wider project which is the conclusion, the testament of this civilization of image: the Didactic Project of Dürer.

The perspective system that emerges during the Renaissance will remain the only one in composition, after a period of cohabitation with sacred geometry - in the same works. Later, the « art of the diagonals », with classical painters, will try to find simple rules, but without finding the lost secrets of the real geometry. And slowly, allegories will take the place of symbols...

We could speak about 20th Century, but I am afraid that after the magnificent fireworks of Renaissance, your eyes will be disappointed, and your mind, as mathematicians, a little bit insulted. I would prefer to expose more elements of geometry with eyes. The must!
Andreï Rublev signs his Holy Trinity by a simple rectangle, instead of the traditional monogram of Christ - or a sentence from the Bible. This sign is not involved in the narrative of the work, it is not more a signature or an ornament. By this simple rectangle, Andrei Rublev claims explicitly for sacred geometry.

This rectangle refers to the golden ratio. Historical detail: *In the early twentieth century, the british critic and fencer Theodore Andrea Cook (1867-1928) agrees with his friend, the American mathematician Mark Barr, to introduce the Greek letter φ as a mathematical symbol of the Golden Ratio - in reference to the greek sculptor Phidias (5th century B.C.).*
The front panel is a lesson of arithmetic values. Beginning by the equation, here in red by the geometry: \[ \varphi^2 = \varphi + 1 \]

The lesson of arithmetic by Andreï Rublev

These purely arithmetical explanations develop in the lower part of the work. And they accord their measure to the (large) width of Rublev's rectangle. In the upper part, however, the arguments are purely geometric - with eyes. Here we find the duality of Horus and his two eyes. That of the archer/geomter opened to aim the target, the other closed to calculate (Oudjat eye). Rublev emphatically defines his great grid with two points named Alpha and Omega.

This icon measures exactly 4 units in width, and Rublev's rectangle 2/7. The arithmetical logic meets the geometrical logic.
Confirmation of the point Omega

Omega is the meeting point of the light. The platform proposes a first invitation. Finally, the Holy Spirit will give his blessing by a 45° line on which he seems to write.

The three angels represent from the left to the right: the Father, the Son and the Holy Spirit.
The two points Alpha and Omega are the key of the grid.

**Remark 1**: the definition of the grid is never an ordinary attempt. As we see through this example, its definition needs a complete set of evidence. This case is very important, because the grid has always been the base of sacred geometry. All the figures and the links need its support to build.

**Remark 2**: it would be impossible to translate the symbolic meaning without any measure. Because of that, the grid is irreplaceable.

**Remark 3**: To understand this art, we need elements of mathematics (Part I).
In the center, the platform creates two lines at an angle of $8^\circ$ and $9^\circ$. $8^\circ$ corresponds to $2\pi/45$, and $9^\circ$ is typical of the logic of the pentagram. The two points of contact with the vertical of Omega are separated by the radius of a circle. Its surface is One. This is **squaring the circle**.

In the low part of the Icon, Rublev exposes arithmetics. In the upper part, he practices geometrical principles. Spiritual/Concrete.

Now, we can really study Trinity and other masterpieces. In Trinity, two angels bow the one who is necessarily the Father. The Christ is at the center in a classical iconographical position - called “enthroned”.

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Yvo Jacquier - ART & GEOMETRY
The translation of this picture could be: From the center of the incircle, the bisector of order one shows the Father, the second (golden) shows the Son, and the third point out the Holy Spirit.

In the other sense from the points to the sides, the first brings unity (1) to human (5), the second brings inspiration/faith (2) to the Earth (4), and the third brings Celestial (3) to the sky (3) in a mirror effect.

The golden point of the inscribed circle is on the hand of the Holy Spirit. And the basic rectangle which takes the measures of the intimus circle (2) and the golden measure (2φ) reaches the top of the frame.

This icon gives the occasion to precise a fundamental aspect of symbolic: symbols don’t live among geometrical figures which communicate through some “analogical thoughts”. These images are a real language, with real structures. So, what is a structure? We could compare with music. Harmony puts several accords one after one. Sacred geometry puts several layers one over one. The links between the different figures is like the ones between different accords. The “Music of the spheres” dear to Plato is not a phantasm. The Grid is the first step of this particular culture. Then, we can approach each figure, each internal line with the measure of this grid.
How to be sure of a composition? How to be secure in the process of study? Science brings answers for a part. The first of them is “margin of precision”. The more you are precise, the closer you are to the truth. We will choose another example to illustrate this aspect, a work which is free of any damage of restoration. A Self-portrait by Dürer - dating 1500. He explains the combinations of golden ratio with the circle $1/\varphi^2$.

**Comparison of different works**

The main problem with geometry, especially with organized systems, is in what I call the “Harmonics” (it is a musical vocabulary). A complex of geometrical forms produces naturally a lot of other forms. But these extra-forms are not in the initial will of the authors.

They come like the heat with the movement.
Just for eyes (with glasses in my case)

Just for brain (with Oudjat eye)

The key-figure of the composition
The birth of Venus - S. Botticelli

The triangle 3-4-5 shows the syntax of its elements, in particular the golden bisector coming to the navel of Venus.
A fabulous construction combines two beautiful fans, two sets of lines separated by the angle of 9°. Second element: a vesica piscis of two circles diameter 5. Third element: two rectangles 3x4 inclined of 27°. Fetish number for Pythagoreans. It means 4 triangles 3-4-5. And the key is that one line of the rectangle finds the point of a symmetric triangle. Very special property, because it is due to the “fan of a pentagram”.

The head of Venus is outside the almond.

*Vesica piscis non caput.*

*Piscis primum a capite foetet.*

(Ryba smrdí od hlavy)

The original title goes in this sense (Vénus anadyomène)

Next page: In the case of « The birth of Venus » by Botticelli, one key of the work is hanged to a kind of shell called Cyprée in France (Il Cyprus). Like Dürer in his self-portrait, Botticelli speaks about the golden logic, but this time by angles. This is a second key in the work.
Golden development
This story in history of art takes 250 page to be explained. I will try to resume but I just finish a book about, accessible from my websites jacquier.org and melencolai.org. Unfortunately, it is still in french...

« DÜRER ET SES TAROTS »

Speech :
Melencolia, the famous work by Albrecht Dürer, 500 years! More famous in its day than the Mona Lisa, it hides more secrets in its lines than in a smile. Melencolia is THE key to a language, heir of Egyptian, Mesopotamian and Greek knowledge. In the Middle Ages this tradition resides at Byzantium until the fall of Constantinople in 1453, which marks the beginning of the Renaissance. Italian artists take up the torch and they undertake an encyclopedia of symbols. This art of composition is practicing the golden ratio of the triangle3-4-5, and the images combine their subjects like a puzzle. It takes an engraver, he will be Dürer. Four prints and a set of cards, "Tarots de Marseille" materialize this project. Melencolia is the portal of a lost civilization, which chose the image to write. Ten years of research in collaboration with scientists and Symbolists, have been necessary to restore this forgotten Culture. And Dürer has provided everything!

English introduction (basic) :
Informations

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Yvo Jacquier
invited by :
Mgr. Zdenek Halas, DiS.
& Ph.D. et PhDr. Alena Šarounová, CSc.

This article

Français
http://www.jacquier.org/Yvo_Jacquier-Geometrie_Sacree.pdf

English

Český
http://www.art-renaissance.net/Charles_University/Yvo_Jacquier_sakralni_geometrie.pdf

Expended version of the mathematical corpus

Français

English

Český